SYDNEY BOYS HIGH SCHOOL
MOORE PARK, SURRY HILLS

2014
HIGHER SCHOOL CERTIFICATE
TRIAL PAPER

Mathematics Extension 2

General Instructions

- Reading Time – 5 Minutes
- Working time – 3 Hours
- Write using black or blue pen. Pencil may be used for diagrams.
- Board approved calculators maybe used.
- Answer Questions 1 to 10 on the sheet provided.
- Each Question from 11 to 16 is to be returned in a separate bundle.
- All necessary working should be shown in every question

Total Marks – 100

- Attempt questions 1 – 16
- Answer in simplest exact form unless otherwise instructed

Examiner: *P.R.Bigelow*

- NOTE: This is a trial paper only and does not necessarily reflect the content or format of the final Higher School Certificate examination paper for this subject.

Use Multiple Choice Answer Sheet

Question 1

Seven people are to be placed in four hotel rooms.
In how many ways may this be done?

A: 4^7

B: ${}^7C_4$

C: ${}^7P_4$

D: 7^4

Question 2

$i^{2114} =$

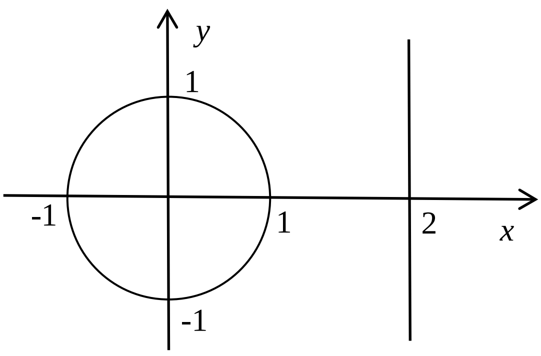
A: 1

B: i

C: $-i$

D: -1

Question 3



The circle $x^2 + y^2 = 1$ is rotated about the line $x = 2$. With use of cylindrical shells, the volume is given by:

A: $4\pi\int_{-1}^{1}(2-x)\sqrt{1-x^2}\,dx$

B: $8\pi\int_{0}^{1}(2-x)\sqrt{1-x^2}\,dx$

C: $2\pi\int_{-1}^{1}(2-x)\sqrt{1-x^2}\,dx$

D: $4\pi\int_{1}^{2}(2-x)\sqrt{1-x^2}\,dx$

Question 4

The equation of the chord of contact from $(5,-2)$ to the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$ is given by:

A: $\frac{x}{8}-\frac{5y}{16}=1$

B: $\frac{5x}{16}+\frac{2y}{9}=1$

C: $\frac{5x}{16}-\frac{2y}{9}=0$

D: $\frac{5x}{16}-\frac{2y}{9}=1$

Question 5

The roots of $x^3+5x+11=0$ are $\alpha,\beta,$ and γ.

The value of $\alpha^2\beta\gamma+\alpha\beta^2\gamma+\alpha\beta\gamma^2$ is:

A: 25

B: 0

C: –55

D: 55

Question 6

If *a* and *b* are positive, which of the following is false?

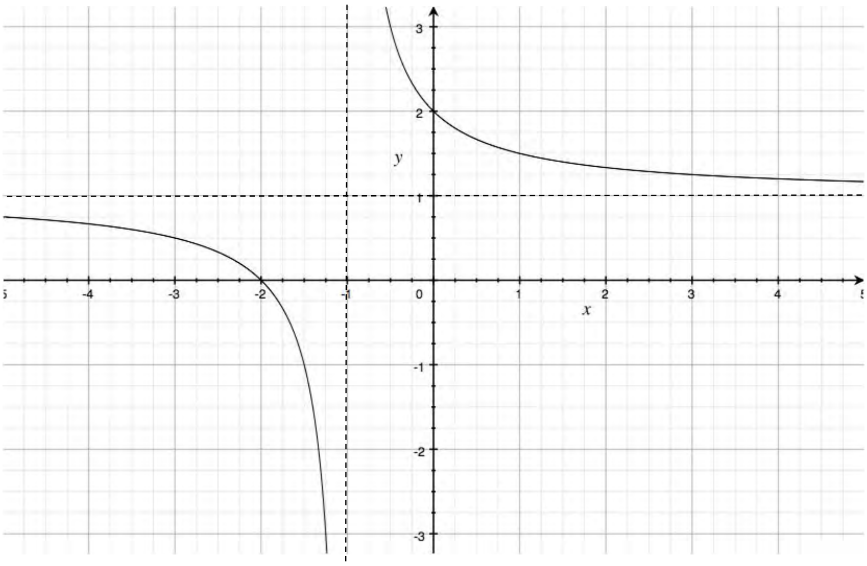
A: $\frac{a}{b}+\frac{b}{a}\geq 2$.

B: $\frac{a+b}{2}\leq\sqrt{ab}$.

C: $\left(\sqrt{a}-\sqrt{b}\right)^2\geq 2ab$.

D: $(a+b)^2\geq(a-b)^2+(2ab)^2$.

Question 7



The graph has equation:

A: $(x-1)(y+1)=1$

B: $y=\dfrac{x+2}{x}$

C: $(x+1)(y-1)=1$

D: $y=\dfrac{x}{x+1}$

Question 8

$1+i$ is a zero of x^3+ax+b where *a*, *b* are real, therefore the values of *a* and *b* are:

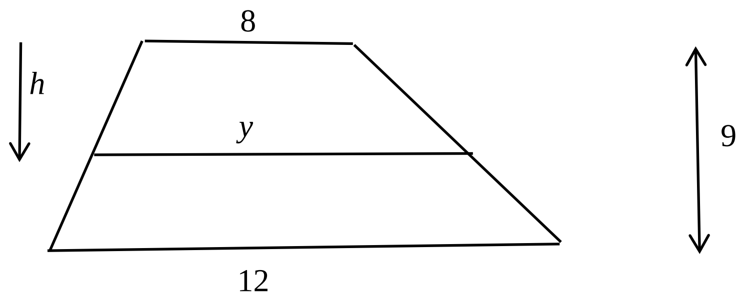
A: $a=-2,\ \ b=-4$

B: $a=-2,\ \ b=4$

C: $a=2,\ \ b=-4$

D: $a=2,\ \ b=4$

Question 9



The diagram shows a trapezium, with an internal parallel line. Which of the following is true?

A: $y = \frac{3}{4}h + 8.$

B: $y = \frac{3}{4}h + 9.$

C: $4y = 9h + 72$

D: $9y = 4h + 72$

Question 10

By considering the graphs of $y = 3x^2 - 2x - 2$ and $y = |3x|$, the solution to $3x^2 - 2x - 2 \leq |3x|$ is:

A: $-\frac{1}{3} \leq x \leq 2.$

B: $-1 \leq x \leq \frac{3}{2}.$

C: $-\frac{1}{3} \leq x \leq \frac{3}{2}$

D: $-1 \leq x \leq 2$

Question 11. (15 marks) **(Start a new answer booklet.)**

Marks

(a) Given $z = 1 - i$, find the values of *w* such that **2**

$$w^2 = i + 3\bar{z}$$

(b) On separate Argand diagrams, shade the following regions:

(i) $4 \le z + \bar{z} \le 10$ **1**

(ii) $\arg\left(z^2\right) = \dfrac{2\pi}{3}$ **1**

(iii) $z\bar{z} = 4$ **1**

(c) (i) Show that $z = 1 + i$ is a root of the polynomial

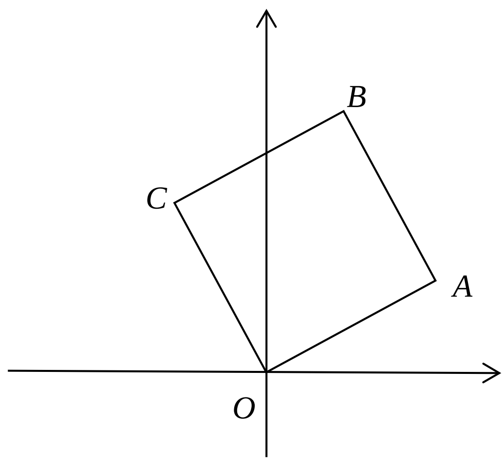
$$z^2 - (3 - 2i)z + (5 - i) = 0$$ **1**

(ii) Find the other root. **1**

(d) *OABC* is a square in the Argand diagram. **3**

B represents the complex number $2 + 2i$.

Find the complex numbers represented by *A* and *C*.



Question 11 (Continued)

(e) From $\{1,2,3,4,5,6,7,8,9\}$ codes of three digits are formed, where no digit is repeated.

(i) Find the number of possible different codes. **1**

(ii) How many of these are *not* in decreasing order of magnitude, reading from left to right? **2**

(f) Given that $\alpha, \beta,$ and γ are the roots of $x^3 - 7x + 6 = 0$, evaluate

$$\alpha^3 + \beta^3 + \gamma^3$$ **2**

Question 12. (15 marks) **(Start a new answer booklet.)**

Marks

(a) Find $\int xe^{4x}\,dx$. **2**

(b) Evaluate $\int_{\frac{3\pi}{2}}^{\frac{5\pi}{2}} \frac{dx}{\cos x + 2}$. **2**

(c) Find $\int \frac{du}{8+u^3}$. **2**

(d) Evaluate $\int_0^{\frac{\pi}{4}} \cos^5\theta\, d\theta$ **2**

(e) (i) Find $\int \frac{dx}{x^2+2x+10}$. **1**

(ii) Hence find $\int \frac{x^2}{x^2+2x+10}dx$. **2**

(f) Consider the curve defined by $2x^2+xy-y^2=0$. **2**

Find the values of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at the point $P(2,4)$.

(g) Sketch the locus $|z-1|+|z+1|=4$(where z is a complex number), showing x and y intercepts. **2**

Question 13. (15 marks) **(Start a new answer booklet.)**

Marks

(a) Find the values of the real numbers *p* and *q* given that **2**

$$x^3 + 2x^2 - 15x - 36 = (x+p)^2(x+q)$$

(b) An ellipse has equation

$$\frac{x^2}{9} + \frac{y^2}{4} = 1$$

(i) Find the eccentricity of the ellipse. **1**

(ii) Sketch the ellipse showing foci, directrices and intercepts. **2**

(iii) Prove that the equation of the tangent to the ellipse at the point $P(3\cos\theta, 2\sin\theta)$ is $2x\cos\theta + 3y\sin\theta = 6$. **3**

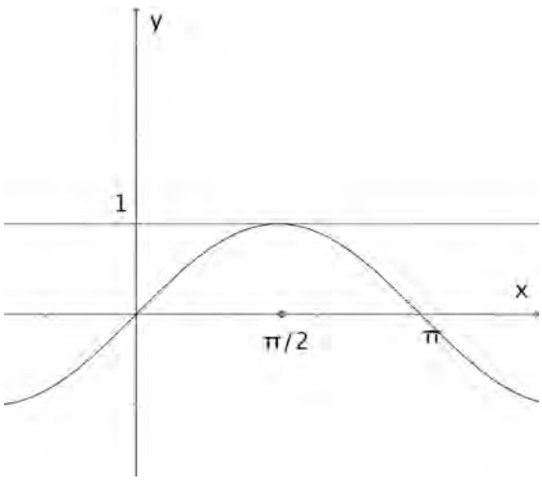
(iv) The ellipse meets the *y*-axis at points *A* and *B*. The tangents to the ellipse at *A* and *B* meet the tangent at *P*, at points *C* and *D* respectively. **3**

Prove that $AC \times BD = 9$.

(c) The area defined by $0 \le x \le \frac{\pi}{2}$, $0 \le y \le 1$ and $y \ge \sin x$ is rotated about the line $y = 1$. **4**

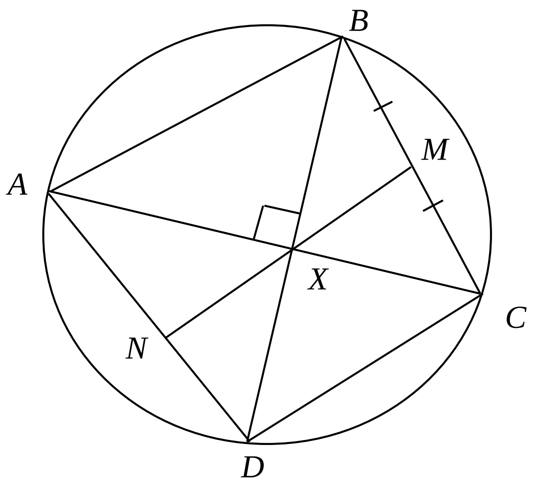
(i) Copy the diagram and shade the defined area.

(ii) Find the volume of the solid by taking slices perpendicular to the axis of rotation.



Question 14 (15 marks) **(Start a new answer booklet.)**

(a) *ABCD* is a cyclic quadrilateral. The diagonals *AC* and *BD* intersect at right-angles at *X*. *M* is the mid-point of *BC*, and *MX* produced meets *AD* at *N*.



(i) Copy the diagram to your answer booklet, then show that $BM = MX$. **1**

(ii) Show that $\angle MBX = \angle MXB$. **1**

(iii) Show that *MN* is perpendicular to *AD*. **3**

(b) The base of a solid is in the shape of an ellipse with equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Vertical cross-sections taken perpendicular to the major axis are rectangles where length is double the height.

(i) Show that the volume of a typical rectangular slice is **2**

$$\delta V = \frac{2b^2}{a^2}\left(a^2 - x^2\right).\delta x$$

(where δx is the width of the slice.)

(ii) Find the volume of the solid by integration. **2**

Question 14 (Continued)

(c) In each of the following parts, $x, y, z, w, a, b, c, d > 0$:

(i) Show that $(x+y)^2 \geq 4xy$. **1**

(ii) Show that $\left[(x+y)(z+w)\right]^2 \geq 16xyzw$ **1**

(iii) Deduce that $\dfrac{x+y+z+w}{4} \geq \sqrt[4]{xyzw}$ **2**

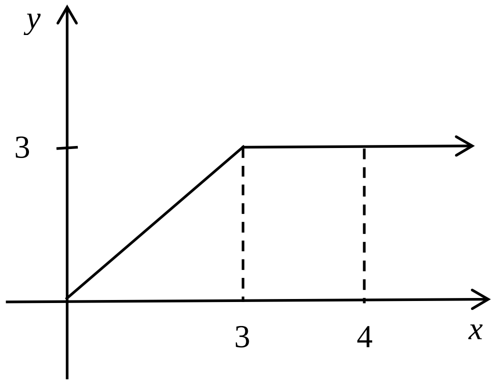
(iv) Hence show that (using (iii)): **2**

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{d}+\frac{d}{a} \geq 4$$

Question 15 (15 marks) **(Start a new answer booklet.)**

Marks

(a) The graph of $y = f(x)$ is shown.



Sketch the following on separate diagrams:

(i) $y = f(4-x)$. **1**

(ii) $y = f(|x|)$. **1**

(iii) $y \times f(x) = 1$. **1**

(iv) $y^2 = f(x)$. **1**

(b) Let w be a non-real cube root of unity.

(i) Show that $1 + w + w^2 = 0$. **1**

(ii) Simplify $(1+w)^2$. **1**

(iii) Show that $(1+w)^3 = -1$. **1**

(iv) Using part (iii) simplify $(1+w)^{3n}$ where $n \in Z^+$. **1**

(v) Show that **3**

$$\binom{3n}{0} - \frac{1}{2}\left[\binom{3n}{1} + \binom{3n}{2}\right] + \binom{3n}{3} - \frac{1}{2}\left[\binom{3n}{4} + \binom{3n}{5}\right] + \binom{3n}{6} - \cdots$$

$$\cdots + \binom{3n}{3n} = (-1)^n$$

[Hint: You may use $\text{Re}(w) = \text{Re}(w^2) = -\frac{1}{2}$]

(c) (i) Show that $\ln(ex) > e^{-x}$ for $x \geq 1$. (Use a diagram.) **1**

(ii) Hence show that $\ln(e^n \times n!) > \dfrac{e^n - 1}{e^n(e-1)}$. **3**

Question 16 (15 marks) **(Start a new answer booklet.)**

(a) A Particle P of unit mass is thrown vertically downwards in a medium where the resistive force is proportional to the velocity.

(i) Taking *downwards as positive*, show that $\ddot{x} = g - kv$ for some $k > 0$. **1**

(ii) Given that the initial speed is U and the particle is thrown from a point T, distant d units above a fixed point O, (taken as the Origin) so that the initial conditions are $v = U$ and $x = -d$. **2**

Show that $v = \frac{g}{k} - \left(\frac{g - kU}{k}\right)e^{-kt}$.

(iii) Hence show that: **2**

$$x = \frac{gt - kd}{k} + \left(\frac{g - kU}{k^2}\right)\left(e^{-kt} - 1\right)$$

(iv) A second identical particle Q is dropped from O, at then same instant that P is thrown down. Use the above results to find expressions for v and x as functions of t, for the particle Q. **2**

(v) The particles collide. Find when this occurs, and find the speed at which they collide **3**

(b) (i) Show that: **1**

$$\sin(2r+1)\theta - \sin(2r-1)\theta = 2\sin\theta\cos 2r\theta$$

(ii) Hence shown that: **2**

$$\sin\theta\sum_{r=1}^{n}\cos 2r\theta = \frac{1}{2}\left\{\sin(2n+1)\theta - \sin\theta\right\}.$$

(iii) Hence evaluate: **2**

$$\sum_{r=1}^{100}\cos^2\frac{r\pi}{100}.$$

This is the end of the paper.

STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1}x^{n+1},\ n \neq -1;\ x \neq 0, \text{if } n < 0$$

$$\int \frac{1}{x} dx = \ln x,\ x > 0$$

$$\int e^{ax} dx = \frac{1}{a}e^{ax},\ a \neq 0$$

$$\int \cos ax\, dx = \frac{1}{a}\sin ax,\ \ a \neq 0$$

$$\int \sin ax\, dx = -\frac{1}{a}\cos ax,\ \ a \neq 0$$

$$\int \sec^2 ax\, dx = \frac{1}{a}\tan ax,$$

$$\int \sec ax \tan ax\, dx = \frac{1}{a}\sec ax,\ \ a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\ \ a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\frac{x}{a},\ \ a > 0,\ \ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln\left(x + \sqrt{x^2 - a^2}\right)\ \ x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE: $\ln x = \log_e x,\ \ x > 0$

Student Number: SOLUTIONS

Mathematics Extension 2
Trial HSC 2014

Select the alternative A, B, C or D that best answers the question. Fill in the response oval completely.

Sample: 2 + 4 = (A) 2 (B) 6 (C) 8 (D) 9

A ○ B ● C ○ D ○

If you think you have made a mistake, put a cross through the incorrect answer and fill in the new answer.

A ● B ☒ C ○ D ○

If you change your mind and have crossed out what you consider to be the correct answer, then indicate the correct answer by writing the word *correct* and drawing an arrow as follows.

A ☒ B ☒ (correct) C ○ D ○

<u>Section I:</u> **Multiple choice answer sheet.**

Completely colour the cell representing your answer. Use black pen.

1. ● B C D

2. A B C ●

3. ● B C D

4. A B C ●

5. A ● C D

6. ● B C D

7. A B ● D

8. A ● C D

9. A B C ●

10. A B C ●

2014 Extension 2 Mathematics Trial HSC:
Solutions— Question 11

11. (a) Given $z = 1 - i$, find the values of w such that **[2]**

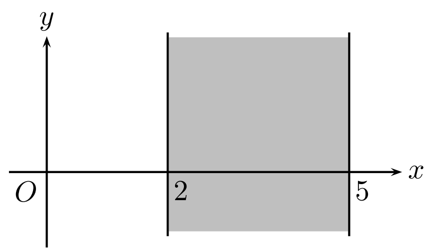
$$w^2 = i + 3\overline{z}.$$

Solution:
$$\begin{aligned} i + 3\overline{z} &= i + 3 + 3i, \\ &= 3 + 4i, \\ &= w^2. \\ \text{Let } w &= a + ib; \\ a^2 - b^2 + 2abi &= 3 + 4i, \\ a^2 - b^2 &= 3, \\ a^2 + b^2 &= 5, \\ ab &= 2, \\ 2a^2 &= 8, \\ a &= \pm 2, \\ b &= \pm 1. \\ \therefore w &= \pm(2 + i). \end{aligned}$$

(b) On separate Argand diagrams, shade the following regions:

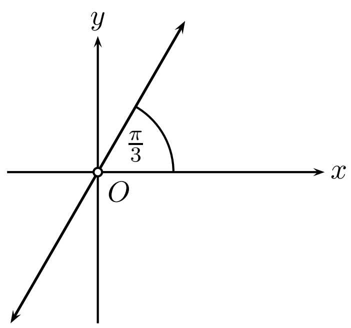
(i) $4 \leqslant z + \overline{z} \leqslant 10$ **[1]**

Solution:
$$\begin{aligned} z + \overline{z} &= 2\Re(z), \\ 4 \leqslant 2\Re(z) &\leqslant 10, \\ 2 \leqslant \Re(z) &\leqslant 5. \end{aligned}$$



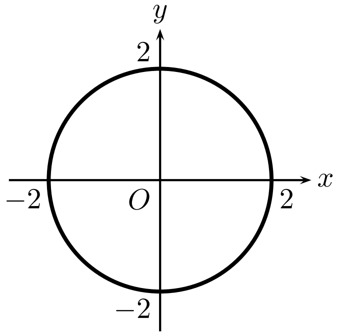
(ii) $\arg\left(z^2\right) = \dfrac{2\pi}{3}$ **[1]**

Solution:
$$\begin{aligned} \arg\left(z^2\right) &= 2\arg(z), \\ &= \frac{2\pi}{3},\ -\frac{4\pi}{3}, \\ \therefore \arg(z) &= \frac{\pi}{3},\ -\frac{2\pi}{3}. \end{aligned}$$



(iii) $z\overline{z} = 4$ [1]

Solution:
$$\begin{aligned} z\overline{z} &= |z|^2, \\ |z| &= 2. \end{aligned}$$



(c) (i) Show that $z = 1 + i$ is a root of the polynomial [1]

$$z^2 - (3 - 2i)z + (5 - i) = 0.$$

Solution:
$$\begin{aligned} \text{Put } P(z) &= z^2 - (3 - 2i)z + (5 - i), \\ P(1 + i) &= (1 + i)^2 - (3 - 2i)(1 + i) + 5 - i, \\ &= 1 + 2i - 1 - (3 + 3i - 2i + 2) + 5 - i, \\ &= i + 5 - 5 - i, \\ &= 0. \end{aligned}$$
i.e. $1 + i$ is a root of $P(z)$.

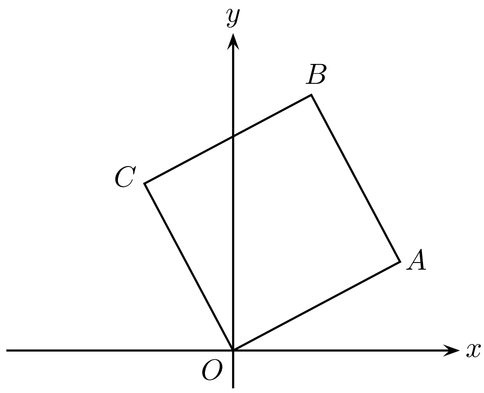
(ii) Find the other root. [1]

Solution: Let the other root be $a + ib$.
$$\begin{aligned} 1 + i + a + ib &= 3 - 2i, \\ 1 + a &= 3, \\ 1 + b &= -2, \\ a &= 2, \\ b &= -3. \end{aligned}$$
$\therefore$ The other root is $2 - 3i$.

(d) $OABC$ is a square in the Argand diagram. [3]

B represents the complex number $2 + 2i$.

Find the complex numbers represented by A and C.



Solution: Method 1—
Notice that $\arg(B) = \frac{\pi}{4}$, so that A must lie on Ox and C must lie on Oy.
Hence $A = (2 + 0i)$ and $C = (0 + 2i)$.

Solution: Method 2—

$$\begin{aligned}
\text{Let } A &= z = a + ib \text{ and so } C = iz = -b + ai.\\
B &= A + C,\\
2 + 2i &= a - b + i(a + b),\\
a - b &= 2,\\
a + b &= 2,\\
2a &= 4,\\
a &= 2,\\
b &= 0.\\
i.e.\ A &= (2 + 0i) \text{ and } C = (0 + 2i).
\end{aligned}$$

Solution: Method 3—

$$\begin{aligned}
|B| &= \sqrt{2^2 + 2^2},\\
&= 2\sqrt{2}.\\
|A| &= 2.\\
A &= \frac{B}{\sqrt{2}} \operatorname{cis}\left(-\tfrac{\pi}{4}\right),\\
&= \frac{2 + 2i}{\sqrt{2}}\left(\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right),\\
&= \frac{2 - 2i + 2i + 2}{\sqrt{2} \times \sqrt{2}},\\
&= 2 + 0i.\\
C &= Ai,\\
&= 0 + 2i.
\end{aligned}$$

(e) From $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ codes of three digits are formed, where no digit is repeated.

(i) Find the number of possible different codes. 1

Solution: $9 \times 8 \times 7 = 504$ different codes.

(ii) How many of these are *not* in decreasing order of magnitude, reading from left to right? 2

Solution: 6 ways of arranging any group of 3, only one of which is in decreasing order of magnitude.
There are ${}^9C_3 = 84$ ways of selecting groups of 3.
Thus there are $5 \times 84 = 420$ which are not decreasing.

(f) Given that α, β, and γ are the roots of $x^3 - 7x + 6 = 0$, evaluate [2]

$$\alpha^3 + \beta^3 + \gamma^3.$$

Solution: Method 1—

$$\begin{aligned}
\alpha + \beta + \gamma &= 0,\\
\alpha\beta + \beta\gamma + \gamma\alpha &= -7,\\
\alpha\beta\gamma &= -6.
\end{aligned}$$

As α, β, and γ are roots,

$$\begin{aligned}
\alpha^3 - 7\alpha + 6 &= 0,\\
\beta^3 - 7\beta + 6 &= 0,\\
\gamma^3 - 7\gamma + 6 &= 0,\\
\alpha^3 + \beta^3 + \gamma^3 - 7(\alpha + \beta + \gamma) + 18 &= 0,\\
\alpha^3 + \beta^3 + \gamma^3 &= -18.
\end{aligned}$$

Solution: Method 2—

$$\begin{aligned}
\text{Put } y &= x^3,\\
x &= y^{\frac{1}{3}}.\\
y - 7y^{\frac{1}{3}} + 6 &= 0,\\
y + 6 &= 7y^{\frac{1}{3}},\\
y^3 + 18y^2 + 108y + 216 &= 343y,\\
y^3 + 18y^2 - 235y + 216 &= 0.\\
\therefore \alpha^3 + \beta^3 + \gamma^3 &= -18.
\end{aligned}$$

Solution: Method 3—

$$\begin{aligned}
(\alpha + \beta + \gamma)^3 &= \big(\alpha^2 + \beta^2 + \gamma^2 + 2\alpha\beta + 2\alpha\gamma + 2\beta\gamma\big)(\alpha + \beta + \gamma),\\
&= \alpha^3 + \alpha\beta^2 + \alpha\gamma^2 + 2\alpha^2\beta + 2\alpha^2\gamma + 2\alpha\beta\gamma + \alpha^2\beta +\\
&\quad \beta^3 + \beta\gamma^2 + 2\alpha\beta^2 + 2\alpha\beta\gamma + 2\beta^2\gamma + \alpha^2\beta + \beta^2\gamma +\\
&\quad \gamma^3 + 2\alpha\beta\gamma + 2\alpha\gamma^2 + 2\beta\gamma^2,\\
&= \alpha^3 + \beta^3 + \gamma^3 + 6\alpha\beta\gamma + 3\big(\alpha\beta^2 + \alpha\gamma^2 + \alpha^2\beta + \beta\gamma^2 +\\
&\quad \alpha^2\gamma + \beta^2\gamma\big),\\
&= \alpha^3 + \beta^3 + \gamma^3 - 3\alpha\beta\gamma + 3(\alpha + \beta + \gamma)(\alpha\beta + \alpha\gamma + \beta\gamma),\\
\alpha^3 + \beta^3 + \gamma^3 &= 3\alpha\beta\gamma + (\alpha + \beta + \gamma)^3 - 3(\alpha + \beta + \gamma)(\alpha\beta + \alpha\gamma + \beta\gamma),\\
&= 3(-6) + 0^3 - 3(0)(-7),\\
&= -18.
\end{aligned}$$

QUESTION TWELVE.

$$\int x e^{4x}\,dx$$

Let $u = x$ $\quad dv = e^{4x}\,dx$

$du = dx$ $\quad v = \frac{1}{4}e^{4x}$

$$\int = \frac{1}{4}xe^{4x} - \frac{1}{4}\int e^{4x}\,dx$$

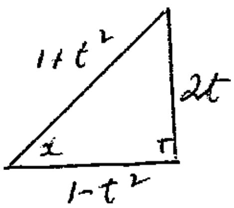
$$= \frac{1}{4}xe^{4x} - \frac{1}{16}e^{4x} + C$$

---

ii) $$\int_{\frac{3\pi}{2}}^{\frac{5\pi}{2}} \frac{dx}{\cos x + 2}$$

Let $t = \tan\frac{x}{2}$

$$\tan x = \frac{2t}{1-t^2}$$



$\cos x = \frac{1-t^2}{1+t^2}$ from diagram

$$\frac{dt}{dx} = \frac{1}{2}\sec^2\frac{x}{2}$$

$$= \frac{1}{2}\left(1 + \tan^2\frac{x}{2}\right)$$

$$= \frac{1}{2}\left(1 + t^2\right)$$

$$\frac{2\,dt}{1+t^2} = dx$$

When $x = \frac{5\pi}{2}$, $t = 1$

$x = \frac{3\pi}{2}$, $t = -1$

$$\int = 2\int \frac{dt}{(1+t^2)\left[\frac{1-t^2}{1+t^2} + 2\right]}$$

$$= 2\int_{-1}^{1} \frac{dt}{1-t^2+2+2t^2}$$

$$= 2\int_{-1}^{1} \frac{dt}{t^2+3}$$

$$= \frac{2}{\sqrt{3}}\tan^{-1}\frac{t}{\sqrt{3}}\Big]_{-1}^{1}$$

$$= \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) + \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

$$= \frac{4}{\sqrt{3}}\cdot\frac{\pi}{6}$$

$$= \frac{2\pi}{3\sqrt{3}} = \frac{2\sqrt{3}\pi}{9}$$

QUESTION 12 c

Consider $\frac{1}{u^3+8} = \frac{A}{u+2} + \frac{Bu+C}{u^2-2u+4}$

$$= \frac{Au^2 - 2Au + 4A + Bu^2 + Cu + 2Bu + 2C}{u^3+8}$$

$$= \frac{u^2(A+B) + u(C+2B-2A) + 4A + 2C}{u^3+8}$$

$A + B = 0$

$2A - 2B - C = 0$

$4A + 2C = 1$

Solving $\Rightarrow$ $A = \frac{1}{12}$, $B = -\frac{1}{12}$, $C = \frac{1}{3}$

Then $\frac{1}{u^3+8} = \frac{\frac{1}{12}}{u+2} + \frac{-\frac{1}{12}u + \frac{1}{3}}{u^2-2u+4}$

$$= \frac{\frac{1}{12}}{u+2} - \frac{1}{12}\left(\frac{u-4}{u^2-2u+4}\right)$$

$$= \frac{\frac{1}{12}}{u+2} - \frac{1}{12}\left[\frac{1}{2}\left(\frac{2u-2}{u^2-2u+4}\right) - \frac{3}{u^2-2u+4}\right]$$

$$= \frac{\frac{1}{12}}{u+2} - \frac{1}{24}\left(\frac{2u-2}{u^2-2u+4}\right) + \frac{1}{4}\left(\frac{1}{(u-1)^2+3}\right)$$

Then

$$\int \frac{du}{u^3+8} = \frac{1}{12}\ln(u+2) - \frac{1}{24}\ln(u^2-2u+4) + \frac{1}{4\sqrt{3}}\tan^{-1}\frac{(u-1)}{\sqrt{3}}$$

TWELVE d.

$$\int_0^{\frac{\pi}{4}} \cos^5\theta \, d\theta = \int \cos^2\theta \cdot \cos^2\theta \cos\theta \, d\theta$$

$$= \int (1-\sin^2\theta)^2 \cos\theta \, d\theta$$

$$= \int (1 - 2\sin^2\theta + \sin^4\theta) \cos\theta . d\theta$$

$$= \sin\theta - \frac{2}{3}\sin^3\theta + \frac{1}{5}\sin^5\theta \Big]_0^{\frac{\pi}{4}}$$

$$= \frac{1}{\sqrt{2}} - \frac{2}{3}\left(\frac{1}{\sqrt{2}}\right)^3 + \frac{1}{5}\left(\frac{1}{\sqrt{2}}\right)^5$$

$$= \frac{1}{\sqrt{2}} - \frac{2}{3} \times \frac{1}{2\sqrt{2}} + \frac{1}{5} \times \frac{1}{4\sqrt{2}}$$

$$= \frac{1}{\sqrt{2}} - \frac{1}{3\sqrt{2}} + \frac{1}{20\sqrt{2}}$$

$$= \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{6} + \frac{\sqrt{2}}{40}$$

~~$= \frac{20\sqrt{2}}{40}$~~

$$= \frac{60\sqrt{2}}{120} - \frac{20\sqrt{2}}{120} + \frac{3\sqrt{2}}{120}$$

$$= \frac{43\sqrt{2}}{120}$$

TWELVE e

$$\int \frac{dx}{x^2+2x+10} = \int \frac{dx}{(x+1)^2+9}$$

$$= \frac{1}{3} \tan^{-1} \frac{x+1}{3} + C$$

AND

$$\frac{x^2}{x^2+2x+10} = \frac{x^2+2x+10}{x^2+2x+10} - \frac{2x+10}{x^2+2x+10}$$

$$= 1 - \frac{2x+2}{x^2+2x+10} - \frac{8}{x^2+2x+10}$$

$$= 1 - \frac{2x+2}{x^2+2x+10} - \frac{8}{(x+1)^2+9}$$

AND

$$\int \frac{x^2 dx}{x^2+2x+10} = x - \ln(x^2+2x+10) - \frac{8}{3} \tan^{-1} \frac{x+1}{3} + C$$

QUESTION 12 f

$$2x^2 + xy - y^2 = 0$$

$$4x + x\frac{dy}{dx} + y - 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx}(2y - x) = 4x + y$$

$$\frac{dy}{dx} = \frac{4x+y}{2y-x}$$

$$= \frac{8+4}{8-2} \quad \text{at } P(2,4)$$

$$= 2$$

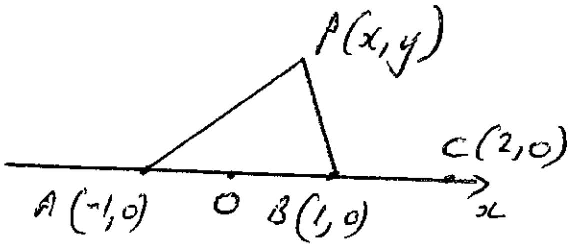
$$\frac{d^2y}{dx^2} = \frac{(2y-x)\left(4+\frac{dy}{dx}\right) - (4x+y)\left(2\frac{dy}{dx} - 1\right)}{(2y-x)^2}$$

$$= \frac{(6)(6) - (12)(3)}{36}$$

$$= 0$$

QUESTION TWELVE g.

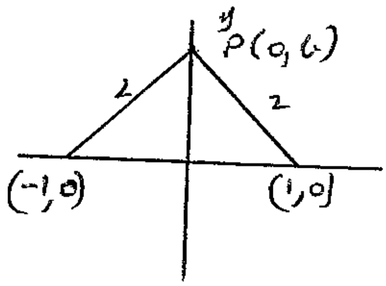
$|z-1| + |z+1| = 4$



$PA + PB = 4$

P has position on the x axis where $y = 0$ of C(2,0)

Hence major axis has length 4



Let P have position (0, b) on the y axis

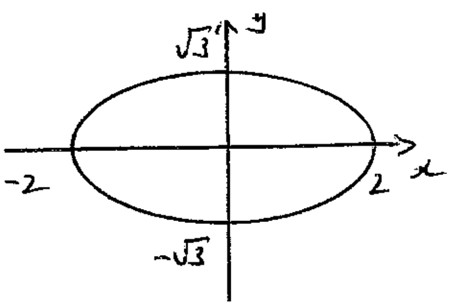
Then

$$\sqrt{b^2+1} + \sqrt{b^2+1} = 4$$

$$\sqrt{b^2+1} = 2$$

$$b^2 + 1 = 4$$

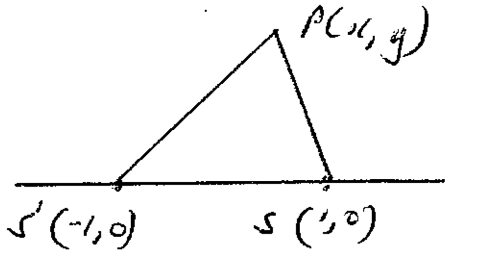
$$b = \sqrt{3}$$



Locus is ellipse

$$\frac{x^2}{4} + \frac{y^2}{3} = 1$$

ALTERNATIVELY USING DISTANCES



$PS' + PS = 4$

$$\sqrt{(x-1)^2 + y^2} + \sqrt{(x+1)^2 + y^2} = 4$$

$$\sqrt{(x-1)^2 + y^2} = 4 - \sqrt{(x+1)^2 + y^2}$$

$$x^2 - 2x + 1 + y^2 = 16 - 8\sqrt{(x+1)^2 + y^2} + x^2 + 2x + 1 + y^2$$

$$-4x - 16 = -8\sqrt{(x+1)^2 + y^2}$$

$$x + 4 = 2\sqrt{(x+1)^2 + y^2}$$

$$\frac{x}{2} + 2 = \sqrt{(x+1)^2 + y^2}$$

$$\frac{x^2}{4} + 2x + 4 = x^2 + 2x + 1 + y^2$$

$$\frac{3}{4}x^2 + y^2 = 3$$

$$\frac{x^2}{4} + \frac{y^2}{3} = 1$$

Q13. EXT-α

(a) $x^3+2x^2-15x-36 = (x+p)^2(x+q)$

"$-p$" is a double root

$f'(x) = 3x^2+4x-15$

Let $f'(x)=0 \Rightarrow 3x^2+4x-15=0$

$(3x-5)(x+3)=0$

$3x \quad -5$

$x \quad 3$

$x = 5/3$ or $x = -3$.

Then $f(-3) = -27+18+45-36 = 0$

$\therefore$ $\underline{p=3}$

2

$\Rightarrow f(x) = (x+3)^2(x+q)$

$\Rightarrow x^3+2x^2-15x-36 = (x^2+6x+9)(x+q)$

By inspection, $\underline{q=-4}$.

$\therefore$ $\underline{p=3}$, $\underline{q=-4}$

(b) $\frac{x^2}{9}+\frac{y^2}{4}=1$

(i) $e = \frac{\sqrt{a^2-b^2}}{a}$

$= \frac{\sqrt{9-4}}{3}$

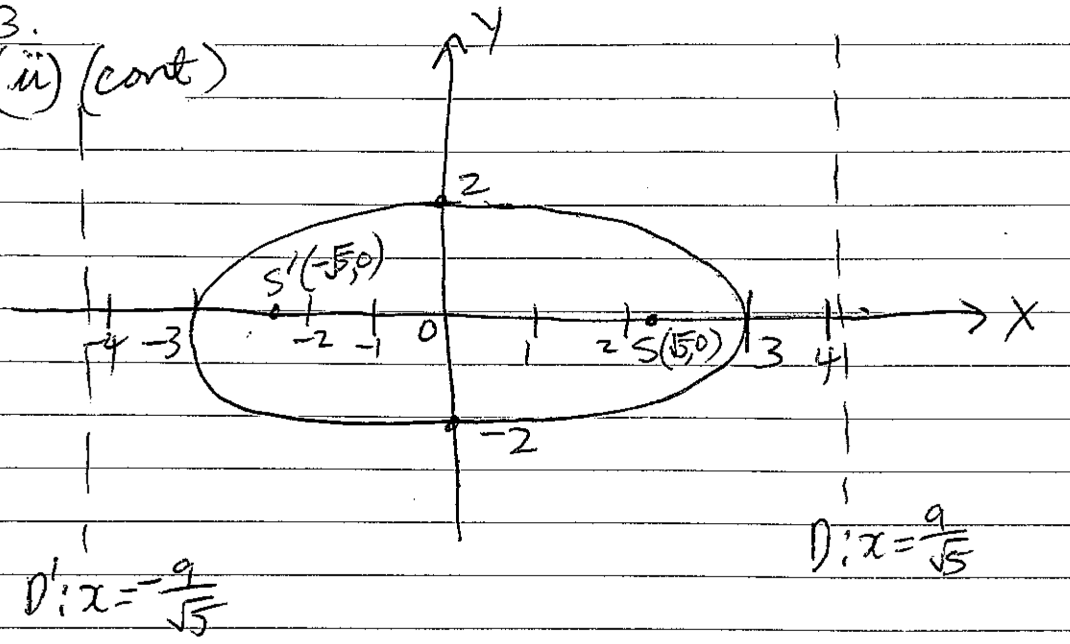
$= \frac{\sqrt{5}}{3}$.

(ii) $S = (\pm ae, 0) = (\pm\sqrt{5}, 0)$

D: $x = \pm\frac{a}{e} \Rightarrow x = \pm\frac{3}{(\sqrt{5}/3)}$.

$\underline{x = \pm\frac{9}{\sqrt{5}}}$

13.
(b)(ii) (cont)



2

(iii) $\frac{x^2}{9} + \frac{y^2}{4} = 1$

$$\Rightarrow \frac{2x}{9} + \frac{y}{2}\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-4x}{9y}$$

At $P(3\cos\theta, 2\sin\theta)$, $\frac{dy}{dx} = \frac{-12\cos\theta}{18\sin\theta}$

$$\frac{dy}{dx} = \frac{-2\cos\theta}{3\sin\theta}$$

Then Eqn of tangent: $y - y_1 = m(x - x_1)$

$$y - 2\sin\theta = \frac{-2\cos\theta}{3\sin\theta}(x - 3\cos\theta)$$

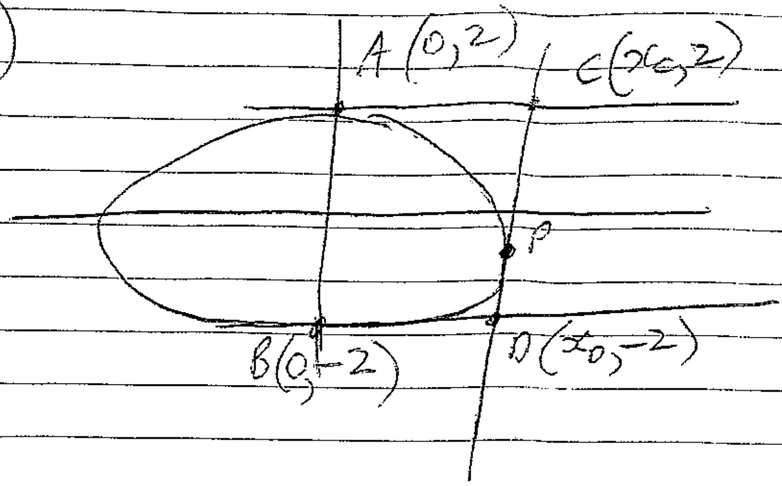
$$3y\sin\theta - 6\sin^2\theta = -2x\cos\theta + 6\cos^2\theta$$

$$\Rightarrow 2x\cos\theta + 3y\sin\theta = 6(\sin^2\theta + \cos^2\theta)$$

$$2x\cos\theta + 3y\sin\theta = 6 \quad \#$$

3

13. (iv)



Eqn of tangent at P : $2x\cos\theta + 3y\sin\theta = 6$ (1)

Eqn tangent at A : $y = 2$ (2)

Eqn tangent at B : $y = -2$ (3)

For C Sub (2) in (1) $\Rightarrow$ $2x\cos\theta + 6\sin\theta = 6$

$$2x\cos\theta = 6(1-\sin\theta)$$

$$x = \frac{6(1-\sin\theta)}{2\cos\theta}$$

$$x_C = \frac{3(1-\sin\theta)}{\cos\theta}$$

$$\therefore C = \left(\frac{3(1-\sin\theta)}{\cos\theta}, 2\right)$$

For D sub (3) in (1) $\Rightarrow$ $2x\cos\theta - 6\sin\theta = 6$

$$2x\cos\theta = 6(1+\sin\theta)$$

$$x_D = \frac{3(1+\sin\theta)}{\cos\theta}$$

$$\therefore D = \left(\frac{3(1+\sin\theta)}{\cos\theta}, -2\right)$$

(3)

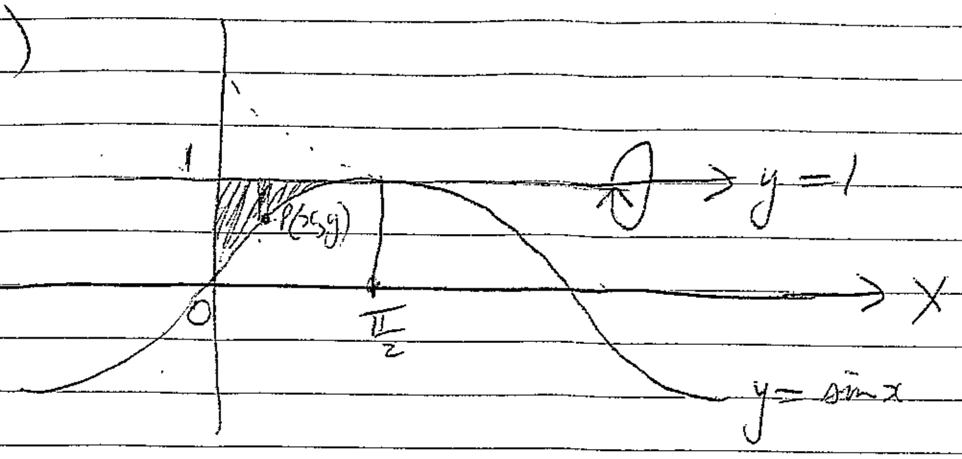
Then $AC = \frac{3(1-\sin\theta)}{\cos\theta}$ and $BD = \frac{3(1+\sin\theta)}{\cos\theta}$

$$\therefore AC \times BD = \frac{3(1-\sin\theta)}{\cos\theta} \times \frac{3(1+\sin\theta)}{\cos\theta} = \frac{9(1-\sin^2\theta)}{\cos^2\theta}$$

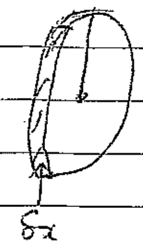
$$= 9 \ \#$$

13. (c)

(i)



(ii) Washers.



$r = 1 - y$

$\Rightarrow r = 1 - \sin x$

$$V_{slice} = \pi r^2 \delta x$$

$$= \pi (1 - \sin x)^2 \delta x$$

$$V = \lim_{\delta x \to 0} \sum_{x=0}^{\pi/2} \pi (1 - \sin x)^2 \delta x$$

$$V = \int_0^{\pi/2} \pi (1 - \sin x)^2 \, dx$$

$$= \pi \int_0^{\pi/2} (1 - 2\sin x + \sin^2 x) \, dx$$

$$= \pi \int_0^{\pi/2} (1 - 2\sin x + \tfrac{1}{2} - \tfrac{1}{2}\cos 2x) \, dx$$

$$= \pi \int_0^{\pi/2} (\tfrac{3}{2} - 2\sin x - \tfrac{1}{2}\cos 2x) \, dx$$

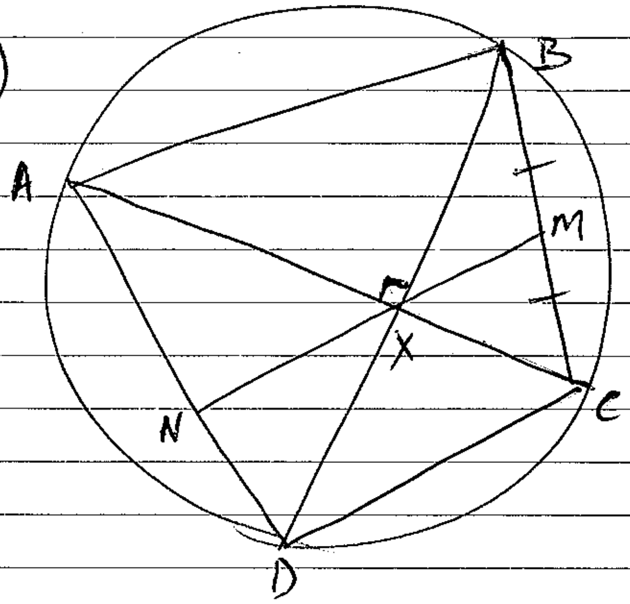
$$= \pi \left[\tfrac{3}{2}x + 2\cos x - \tfrac{1}{4}\sin 2x \right]_0^{\pi/2}$$

$$= \pi \left[\left(\tfrac{3\pi}{4} + 0 - 0\right) - (0 + 2 - 0) \right]$$

$$= \left(\frac{3\pi^2}{4} - 2\pi \right) \text{units}^3.$$

4

14) a) i)

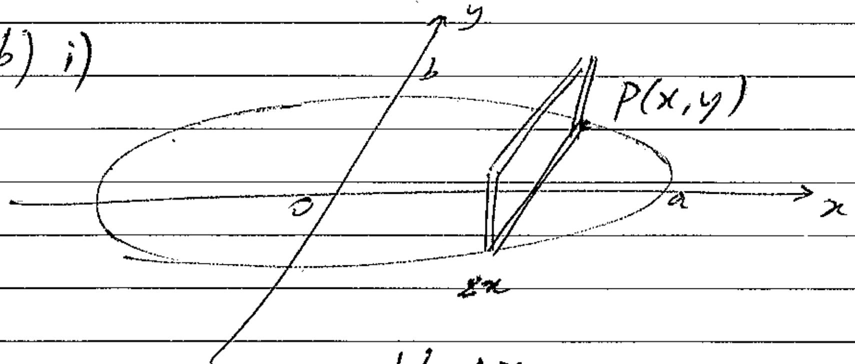


since $\angle BXC = 90^\circ$
BC is the diameter of circle BCX
since M is the midpoint of B
M is the centre of circle BCX
$BM = MX$ (equal radii)

ii) $\triangle MBX$ is isosceles ($BM = MX$)
$\therefore \angle MBX = \angle MXB$ (base angles of isosceles triangle)

iii) Let $\angle MBX = \angle MXB = \alpha$
$\angle BCX = 90 - \alpha$ (angle sum of $\triangle BCX$)
$\angle BDN = 90 - \alpha$ (angles in the same segment)
$\angle NXD = \alpha$ (vertically opposite angles)
$\angle XND = 90^\circ$ (angle sum of $\triangle NXD$)
$\therefore MN \perp AD$

b) i)



$\Delta V = bh\,\Delta x$
$\Delta V = (2y)(y)\Delta x$
$= 2y^2 \Delta x$
$= 2\frac{b^2}{a^2}(a^2 - x^2)\,\Delta x$

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$
$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2}$
$y^2 = b^2\left(1 - \frac{x^2}{a^2}\right)$
$y^2 = \frac{b^2}{a^2}(a^2 - x^2)$

ii) $V = \lim_{\Delta x \to 0} \sum_{x=-a}^{a} \frac{2b^2}{a^2}(a^2 - x^2)\Delta x$

$= \frac{2b^2}{a^2}\int_{-a}^{a}(a^2 - x^2)\,dx$

$= \frac{4b^2}{a^2}\int_{0}^{a}(a^2 - x^2)\,dx$

$= \frac{4b^2}{a^2}\left[a^2x - \frac{x^3}{3}\right]_0^a$

$= \frac{4b^2}{a^2}\left[a^2(a) - \frac{(a)^3}{3} - (0)\right]$

$= \frac{4b^2}{a^2}\left[\frac{2a^3}{3}\right]$

$= \frac{8ab^2}{3}$ cubic units.

c) i) $(x+y)^2 = (x-y)^2 + 4xy$

$\geq 4xy$

ii) similarly $(z+w)^2 \geq 4zw$

$(x+y)^2(z+w)^2 \geq 4xy \cdot 4zw$

$= 16xyzw$

$\therefore \left[(x+y)(z+w)\right]^2 \geq 16xyzw$

iii) From (i) $\left(\left(\frac{x+y}{4}\right) + \left(\frac{z+w}{4}\right)\right)^2 \geq 4\left(\frac{x+y}{4}\right)\left(\frac{z+w}{4}\right)$

$\left(\frac{x+y+z+w}{4}\right)^2 \geq \frac{(x+y)(z+w)}{4}$

$\left(\frac{x+y+z+w}{4}\right)^4 \geq \frac{\left[(x+y)(z+w)\right]^2}{16}$

From (ii)

$$\left(\frac{x+y+z+w}{4}\right)^4 \geq \frac{16xyzw}{16}$$

$$\frac{x+y+z+w}{4} \geq \sqrt[4]{xyzw}$$

iv) let $x=\frac{a}{b}$, $y=\frac{b}{c}$, $z=\frac{c}{d}$, $w=\frac{d}{a}$

$$\frac{\frac{a}{b}+\frac{b}{c}+\frac{c}{d}+\frac{d}{a}}{4} \geq \sqrt[4]{\frac{a}{b}\cdot\frac{b}{c}\cdot\frac{c}{d}\cdot\frac{d}{a}}$$

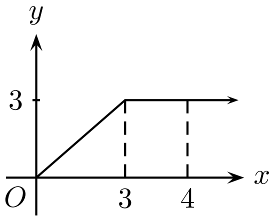
$$\frac{\frac{a}{b}+\frac{b}{c}+\frac{c}{d}+\frac{d}{a}}{4} \geq 1$$

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{d}+\frac{d}{a} \geq 4$$

2014 Extension 2 Mathematics Trial HSC:
Solutions— Question 15

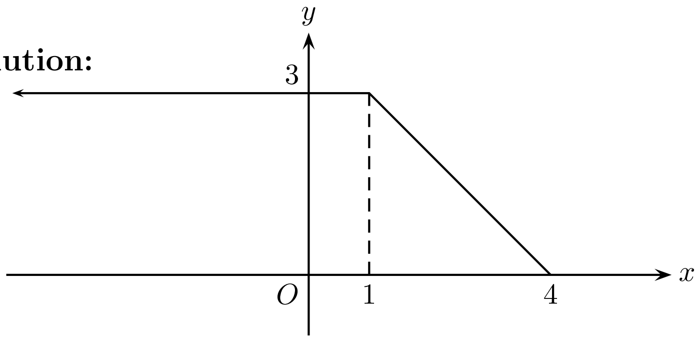
15. (a) The graph of $y = f(x)$ is shown.

Sketch the following on separate diagrams:



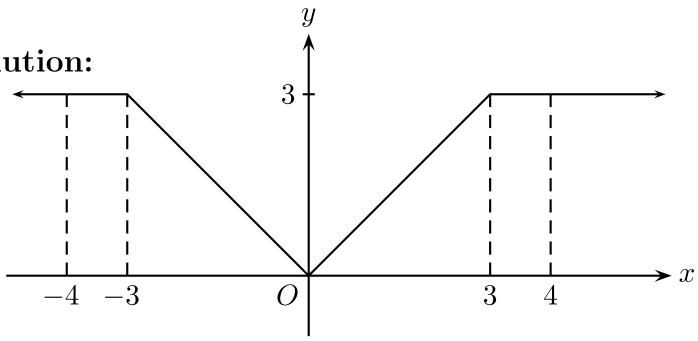
(i) $y = f(4 - x)$, 1

Solution:



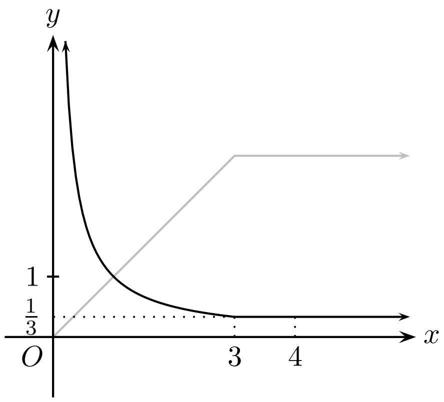
(ii) $y = f(|x|)$, 1

Solution:



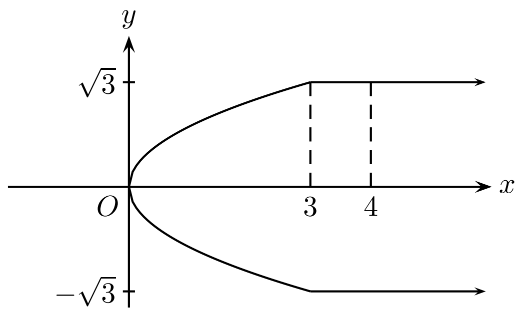
(iii) $y \times f(x) = 1$, 1

Solution:



(iv) y^2= f(x). 1

Solution:



(b) Let w be a non-real cube root of unity.

(i) Show that $1 + w + w^2 = 0$. 1

Solution:
$$\begin{aligned} w^3 &= 1, \\ w^3 - 1 &= 0, \\ (w-1)(w^2+w+1) &= 0, \\ \text{but } w &\neq 1 \text{ as } w \text{ not real}, \\ \therefore w^2 + w + 1 &= 0. \end{aligned}$$

(ii) Simplify $(1 + w)^2$. 1

Solution:
$$\begin{aligned} (1+w)^2 &= w^2 + 2w + 1, \\ &= (w^2 + w + 1) + w, \\ &= w. \end{aligned}$$

(iii) Show that $(1 + w)^3 = -1$. 1

Solution:
$$\begin{aligned} (1+w)^2(1+w) &= w(1+w), \\ &= w + w^2, \\ &= (1 + w + w^2) - 1, \\ &= -1. \end{aligned}$$

(iv) Using part (iii) simplify $(1 + w)^{3n}$ where $n \in \mathbb{Z}^+$. 1

Solution:
$$\begin{aligned} \left((1+w)^3\right)^n &= (-1)^n, \\ &= \begin{cases} -1 & \text{if } n \text{ is odd}, \\ 1 & \text{if } n \text{ is even}. \end{cases} \end{aligned}$$

(v) Show that 3

$$\binom{3n}{0} - \frac{1}{2}\left[\binom{3n}{1} + \binom{3n}{2}\right] + \binom{3n}{3} - \frac{1}{2}\left[\binom{3n}{4} + \binom{3n}{5}\right] + \binom{3n}{6} - \dots$$
$$\dots + \binom{3n}{3n} = (-1)^n$$

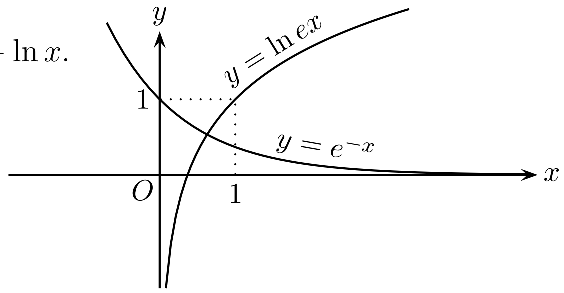
$\left[\text{Hint: You may use } \Re(w) = \Re(w^2) = -\dfrac{1}{2}\right]$.

Solution: Now from part (iv), $(1+w)^{3n} = (-1)^n \in \mathbb{R}$,
so when looking at the expansion of $(1+w)^{3n}$
we need only consider the real parts.
We also note that $w^{3k} = 1$ as $w^3 = 1$, $w^{3k+1} = w$, $w^{3k+2} = w^2$ and,
using $\Re(w) = \Re(w^2) = -\frac{1}{2}$, we have

$$\begin{aligned}(1+w)^{3n} &= \tbinom{3n}{0} + \tbinom{3n}{1}w + \tbinom{3n}{2}w^2 + \tbinom{3n}{3}w^3 + \tbinom{3n}{4}w^4 + \tbinom{3n}{5}w^5 + \dots \\ &\quad \dots + \tbinom{3n}{3n-2}w^{3n-2} + \tbinom{3n}{3n-1}w^{3n-1} + \tbinom{3n}{3n}w^{3n}, \\ i.e.\ (-1)^n &= \tbinom{3n}{0} - \tfrac{1}{2}\big[\tbinom{3n}{1} + \tbinom{3n}{2}\big] + \tbinom{3n}{3} - \tfrac{1}{2}\big[\tbinom{3n}{4} + \tbinom{3n}{5}\big] + \tbinom{3n}{6} - \dots \\ &\quad \dots + \tbinom{3n}{3n}.\end{aligned}$$

(c) (i) Show that $\ln(ex) > e^{-x}$ for $x \geqslant 1$. (Use a diagram.) 1

Solution: $\ln ex = 1 + \ln x$.



(ii) Hence show that $\ln(e^n \times n!) > \dfrac{e^n - 1}{e^n(e-1)}$. 3

Solution: Method 1—
From part (i), $\ln(ex) > e^{-x}$;

$$\begin{aligned}\text{so L.H.S.} &= \ln(1 \times e) + \ln(2e) + \ln(3e) + \dots + \ln\big((n-1)e\big) + \ln(ne), \\ &> e^{-1} + e^{-2} + e^{-3} + \dots + e^{1-n} + e^{-n}, \\ &> \frac{1}{e^n}\big(e^{n-1} + e^{n-2} + \dots + e^2 + e^1 + e^0\big), \\ &> \frac{1}{e^n} \times \frac{e^n - 1}{e - 1}.\end{aligned}$$

$i.e.$ $\ln(e^n \times n!) > \dfrac{e^n - 1}{e^n(e-1)}$.

Solution: Method 2—
Test $n = 1$,

$$\begin{aligned}\text{L.H.S.} &= \ln e, & \text{R.H.S.} &= \frac{e-1}{e(e-1)}, \\ &= 1. & &= \frac{1}{e}.\end{aligned}$$

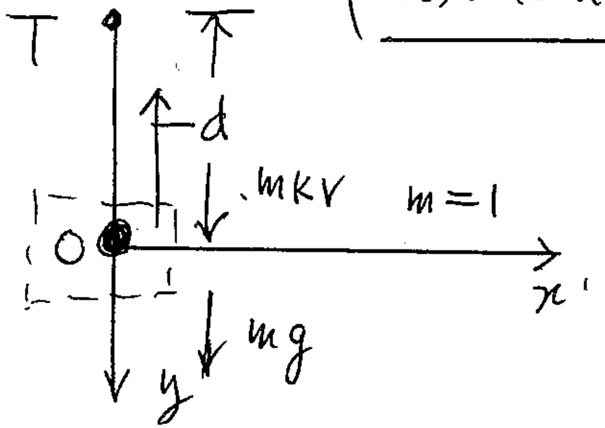
So it is true for $n = 1$.
Now assume true for some $n = k$, $k \in \mathbb{Z}^+$,
$i.e.$ $\ln(e^k \times k!) > \dfrac{e^k - 1}{e^k(e-1)}$.

Then test for $n = k+1$, *i.e.* $\ln\left(e^{k+1} \times (k+1)!\right) > \dfrac{e^{k+1}-1}{e^{k+1}(e-1)}$.

$$\begin{aligned}
\text{L.H.S.} &= \ln\left(e^k.k! \times e(k+1)\right),\\
&= \ln\left(e^k \times k!\right) + \ln\left(e(k+1)\right).\\
\text{Now } \ln(e^k \times k!) &> \frac{e^k-1}{e^k(e-1)} \text{ from the assumption,}\\
\text{and } \ln\left(e(k+1)\right) &> e^{-(k+1)} \text{ from part (i) where } x \geqslant 1,\\
\therefore \text{L.H.S.} &> \frac{e^k-1}{e^k(e-1)} \times \frac{e}{e} + \frac{1}{e^{k+1}} \times \frac{e-1}{e-1},\\
&> \frac{e^{k+1}-e+e-1}{e^{k+1}(e-1)},\\
&> \frac{e^{k+1}-1}{e^{k+1}(e-1)} = \text{R.H.S.}
\end{aligned}$$

Thus true for $n = k+1$ if true for $n = k$, but true for $n = 1$ so true for $n = 2, 3, 4$, and so on for all $n \in \mathbb{Z}^+$.

Question (16).



(i) $\ddot{x} = g - kv$ [1]

(ii) $\dfrac{dv}{dt} = g - kv$.

$$\frac{dt}{dv} = \frac{1}{g - kv}$$

$$\therefore t = -\frac{1}{k}\int\left(\frac{-k}{g - kv}\right)dv$$

$$t = -\frac{1}{k}\ln(g - kv) + c_1$$

When $t = 0$, $v = U$.

$$\therefore 0 = -\frac{1}{k}\ln(g - kU) + c_1$$

$$\therefore c_1 = \frac{1}{k}\ln(g - kU),$$

i.e. $t = \dfrac{1}{k}\ln\left(\dfrac{g - kU}{g - kv}\right)$, [1]

$$\therefore e^{-kt} = \frac{g - kv}{g - kU}$$

$$\therefore g - kv = (g - kU)e^{-kt}$$

$$\therefore kv = g - (g - kU)e^{-kt}$$ [1]

$$\boxed{V_p = \frac{g}{k} - \frac{g - kU}{k}e^{-kt}} \quad \text{—— (1)}$$

(iii) $\dfrac{dx}{dt} = \dfrac{g}{k} - \left(\dfrac{g - kU}{k}\right)e^{-kt}$

$$x = \frac{gt}{k} + \frac{(g - kU)}{k^2}\int(-k)e^{-kt}\,dt$$

$$x = \frac{g}{k}t + \left(\frac{g - kU}{k^2}\right)e^{-kt} + c_2.$$

When $t = 0$, $x = -d$.

$$\therefore -d = \frac{g - kU}{k^2} + c_2$$ [2]

$$\Longrightarrow c_2 = -\left(\frac{g - kU}{k^2}\right) - d.$$

$$\therefore \boxed{x_p = \left(\frac{gt - kd}{k}\right) + \frac{g - kU}{k^2}\left(e^{-kt} - 1\right)} \quad \text{—— (2)}$$

(iv) ——— (1)

$$v_p = \frac{g}{k} - \frac{g-kU}{k} e^{-kt}$$

$$x_p = \left(\frac{gt - kd}{k}\right) + \frac{g-kU}{k^2}\left(e^{-kt} - 1\right)$$

——— (2)

put $U = 0$ [1]

$$\therefore v_q = \frac{g}{k}\left(1 - e^{-kt}\right)$$

——— (3)

put $U = 0,\ d = 0$

$$\Rightarrow x_q = \frac{gt}{k} + \frac{g}{k^2}\left(e^{-kt} - 1\right)$$

——— (4) [1]

(v) The particles collide When $x_p = x_q$

$$\frac{gt - kd}{k} + \left(\frac{g-kU}{k^2}\right)\left(e^{-kt} - 1\right) = \frac{gt}{k} + \frac{g}{k^2}\left(e^{-kt} - 1\right)$$

i.e. $\frac{U}{k} e^{-kt} = \frac{U}{k} - d$

$$e^{-kt} = 1 - \frac{kd}{U}$$

$$-kt = \ln\left(1 - \frac{kd}{U}\right)$$

$$t = -\frac{1}{k}\ln\left(1 - \frac{kd}{U}\right)$$ [1]

——— (5)

When $t = -\frac{1}{k}\ln\left(\frac{U}{U-kd}\right)$

$$v_p = \frac{g}{k} - \frac{g}{k} + U + (g - kU)\frac{d}{U}$$

$$= U + \frac{gd}{U} - kd$$ ——— (6a)

$$v_q = \frac{g}{k}\left(1 - 1 + \frac{kd}{U}\right) = \frac{gd}{U}$$ [2]

——— (6b)

$\therefore$ Speed of collision $|v_p - v_q|$

$$= \left|\left(U + \frac{gd}{U} - kd\right) - \frac{gd}{U}\right|$$

$$= |U - kd|$$ ——— (7)

Question 16 (b)

(i) $\sin(2r+1)\theta - \sin(2r-1)\theta$

$= \sin 2r\theta \cos\theta + \cos 2r\theta \sin\theta$

$- (\sin 2r\theta \cos\theta - \cos 2r\theta \sin\theta)$

$= 2\sin\theta \cos 2r\theta$. [1]

OR

$$\sin A - \sin B = 2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$$

$\sin(2r+1)\theta - \sin(2r-1)\theta$

$= 2\cos\left[\frac{(2r+1)\theta + (2r-1)\theta}{2}\right]\sin\left[\frac{(2r+1)\theta - (2r-1)\theta}{2}\right]$

$= 2\cos\left(\frac{4r\theta}{2}\right)\sin\left(\frac{2\theta}{2}\right)$

$= 2\cos(2r\theta)\sin\theta$ —— (1)

[1]

(ii) ~~From (i)~~

$2\sin\theta \sum_{r=1}^{n}\cos(2r\theta) = \sum_{r=1}^{n}\left[\sin(2r+1)\theta - \sin(2r-1)\theta\right]$

$\therefore$

$\sin\theta \sum_{r=1}^{n}\cos(2r\theta) = \frac{1}{2}\sum_{r=1}^{n}\left[\sin(2r+1)\theta - \sin(2r-1)\theta\right]$

$= \frac{1}{2}\left[(\sin 3\theta - \sin\theta) + (\sin 5\theta - \sin 3\theta) + \cdots + \sin(2n+1)\theta - \sin(2n-1)\theta\right]$

$= \frac{1}{2}\left[\sin(2n+1)\theta - \sin\theta\right]$. [1]

(iii)

$$\cos^2 r\left(\frac{\pi}{100}\right) = \frac{1+\cos\left[2r\left(\frac{\pi}{100}\right)\right]}{2}$$

$\therefore \cos 2r\left(\frac{\pi}{100}\right) = \frac{1}{2} + \frac{1}{2}\cos\left(\frac{2r\pi}{100}\right)$

$$\therefore \sum_{r=1}^{n}\cos 2r\theta = \frac{1}{2\sin\theta}\left[\sin(2n+1)\theta - \sin\theta\right] \quad \text{—— (2)}$$

$\therefore \sum_{r=1}^{100}\cos^2\left(\frac{r\pi}{100}\right) = \sum_{r=1}^{100}\left[\frac{1}{2} + \frac{1}{2}\cos\left(2r\left(\frac{\pi}{100}\right)\right)\right]$

$= \frac{1}{2}\times 100 + \frac{1}{2}\sum_{r=1}^{100}\cos\left(2r\left(\frac{\pi}{100}\right)\right)$ [1]

Using (2) $\theta = \frac{\pi}{100}$, $n = 100$

$\therefore \sum_{r=1}^{100}\cos^2\left(\frac{r\pi}{100}\right) = 50 + \frac{1}{2}\left[\frac{\sin\left(\frac{201\pi}{100}\right) - \sin\frac{\pi}{100}}{2\sin\left(\frac{\pi}{100}\right)}\right]$

$= 50 + \frac{\sin\left(2\pi + \frac{\pi}{100}\right) - \sin\left(\frac{\pi}{100}\right)}{4\sin\left(\frac{\pi}{100}\right)}$ [1]

$= 50 + \frac{\left(\sin\frac{\pi}{100} - \sin\frac{\pi}{100}\right)}{4\sin\frac{\pi}{100}}$ $\left[\sin(\theta + 2\pi) = \sin\theta\right]$

$= 50$